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COMMUTATIVE RINGS AND ZERO-DIVISOR SEMIGROUPS OF SIMPLE GRAPH $K_n - K_2$

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Abstract

In this paper, we study commutative rings and commutative zero-divisor semigroups determined by graphs. We prove that for $n \geq 3$, the graph $K_n - K_2$, a complete graph K_n deleted an edge has corresponding semigroups and the graph $K_n - K_2$ has corresponding rings, if and only if $n = 3$. We obtain a formula $\mathrm{H}(n)$ to calculate the number of non-isomorphic zero-divisor semigroups corresponding to the graph $K_n - K_2$ and by using of a computing programme, the values of $\mathrm{H}(n)$ are listed for $n \leq 100$.

1. Introduction

Given a commutative ring R with multiplicative identity 1 (or a commutative semigroup with zero element 0), recall that the zero-divisor graph of R is the undirected graph, where the vertices are the nonzero

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zero-divisors of R, and where there is an edge between two distinct vertices x and y, if and only if $xy = 0$. The zero-divisor graph of R is denoted by $\Gamma(R)$. This definition of $\Gamma(R)$ first appeared in [2] (in [3] for semigroup case), where some fundamental properties and possible structures of $\Gamma(R)$ were studied. For example, $\Gamma(R)$ is always a simple, connected, and undirected graph with diameter less than or equal to three. For a given connected simple graph G, if there exists a commutative ring (or a commutative semigroup) R such that $\Gamma(R) \cong G$, then we say that G has *corresponding rings* (or *corresponding semigroups*), and we call R a *ring* (a *semigroup*) determined by the graph G. Clearly, if a simple graph G has corresponding rings, then it has corresponding semigroups too.

For any commutative semigroup S, let T be the set of all zero-divisors of S. Then T is an ideal of S and in particular, it is also a semigroup with the property that all elements of T are zero-divisors. We call such semigroups are *zero-divisor semigroups*. Obviously, we have $\Gamma(S) \cong \Gamma(T)$.

Zero-divisor graphs of commutative rings or commutative semigroups were studied in several articles, such as [1, 3, 4, 7, 8, 9, 10, 11]. In [10], Wu and Cheng obtained a formula $\mathrm{K}(n)$ to calculate the number of non-isomorphic zero-divisor semigroups corresponding to the complete graph K_n.

In this paper, we study commutative rings and commutative zero-divisor semigroups determined by graphs. In [9, 10, 11], Wu and his collaborators investigated the zero-divisor semigroups of the complete graph and the complete graph added end vertices. In this paper, we primarily consider the commutative rings and zero-divisor semigroups of the complete graph deleted an edge. We prove that a complete graph deleted an edge has corresponding semigroups and has corresponding rings, if and only if $n = 3$. Then, we get a formula $\mathrm{H}(n)$ to calculate the number of non-isomorphic zero-divisor semigroups corresponding to the graph $K_n - K_2$, the complete graph K_n deleted an edge.

Throughout this paper, all rings are commutative rings with multiplicative identity 1, all semigroups are multiplicative commutative zero-divisor semigroups with zero element 0, where $0x = 0$ for all $x \in S$, and all graphs are undirected simple and connected. Let R be a ring. Recall that the set of zero-divisors $Z(R)$ is the set $\{x \in R|$ there exists $0 \neq y \in R$ such that $xy = 0\}$, and the annihilator of a zero-divisor x is $\mathrm{Ann}(x) = \{y \in Z(R) | xy = 0\}$. For any vertices x, y in a graph G, if x and y are adjacent, we denote it as $x - y$. For other graph theoretical notions and notations adopted in this paper, please refer to [5].

2. Main Results

If M_n is a commutative zero-divisor semigroup with $\Gamma(M_n) \cong K_n - K_2$, the complete graph K_n deleted an edge, we let $M_n = \{0, a_1, \ldots, a_n\}$, and we always assume $K_2 = a_1 - a_2$. Then, we have the following necessary requirements for M_n:

(1) $a_1a_i = 0$ and $a_2a_i = 0$, for all $3 \leq i \leq n$;

(2) $a_ia_j = 0$, for all $i, j \geq 3, i \neq j$;

(3) $a_i^2 \neq a_1, a_2$, for any $3 \leq i \leq n$.

Theorem 2.1. *Let* $n > 1$ *be an integer. Then,*

(1) *There are corresponding rings to the graph* $K_n - K_2$, *if and only if* $n = 3$. *Moreover, the all corresponding rings of graph* $K_3 - K_2$ *are* Z_6, Z_8, *and* $Z_2[X]/(X^3)$.

(2) *There are corresponding semigroups to the graph* $K_n - K_2$, *if and only if* $n \geq 3$.

Proof. If $n = 2$, then the graph obtained by deleting an edge from the complete graph K_2 is not connected, so it has no corresponding rings and no corresponding semigroups.

(1) Assume that $n \geq 4$. If there is a ring R, such that $\Gamma(R) \cong K_n - K_2$, then $Z(R) = \{0, a_1, a_2, \ldots, a_n\}$. Let us first consider $a_1 + a_3$. Clearly, $a_1 + a_3 \in \mathrm{Ann}(a_4) \setminus \mathrm{Ann}(a_2) \subseteq \{a_1, a_2\}$. If $a_1 + a_3 = a_1$, then $a_3 = 0$, a contradiction. Therefore, $a_1 + a_3 = a_2$. Similarly, $a_1 + a_4 = a_2$. Then $a_3 = a_4$, a contradiction too. Hence, there are not corresponding rings to the graph $K_n - K_2$, for all $n \geq 4$. If $n = 3$, the graph $K_3 - K_2$ is $a_1 - a_3 - a_2$, so the corresponding rings of graph $K_3 - K_2$ are Z_6, Z_8, or $Z_2[X]/(X^3)$ by [2, Example 2.1(a)].

(2) If $n \geq 3$, we let $S = V(K_n) \cup \{0\} = \{0, a_1, a_2, \ldots, a_n\}$, and $e = a_1 - a_2$ be the deleted edge. Since a_3 is adjacent to $a_1, a_2, a_4, \ldots, a_n$, by [4, Theorem 2], there exists a commutative zero-divisor semigroup corresponding to $K_n - K_2$. In fact, we can define an operation on S by $a_1^2 = a_1$, $a_2^2 = a_2$, $a_1 a_2 = a_1$, $a_1 a_j = 0$, $a_2 a_j = 0$, $a_i a_j = 0$, and $a_i^2 = 0$, for all $i \geq 3$. It is easy to verify that the operation is associative. This completes our proof. □

In the next, we use $\mathrm{P}(n)$, $\mathrm{K}(n)$, and $\mathrm{H}(n)$ to denote the number of partitions of the integer n, the number of non-isomorphic zero-divisor semigroups corresponding to the complete graph K_n and the number of non-isomorphic zero-divisor semigroups corresponding to the graph $K_n - K_2$, respectively. In [10], Wu and Cheng gave a formula $\mathrm{K}(n)$ to calculate the number of non-isomorphic zero-divisor semigroups corresponding to the complete graph K_n.

Lemma 2.2 [10, Theorem 2.2]**.** *The number of non-isomorphic zero-divisor semigroups corresponding to the complete graph* K_n *is*

$$\mathrm{K}(n) = 1 + \sum_{k=1}^{n} \sum_{t=0}^{n-k} p(n-t,\ k),$$

where $p(j,\ i)$ *is the number of the following partitions of the integer* j:

$$d_1 + d_2 + \cdots + d_i = j,$$

where $1 \leq d_1 \leq d_2 \leq \cdots \leq d_i$. □

In [6], $\mathrm{P}(n)$ denotes the number of partitions of n and the values of $\mathrm{P}(n)$ are listed in the table below for $0 \le n \le 100$. By using values of $\mathrm{P}(n)$, we can simplify the above formula to $\mathrm{K}(n) = \sum_{k=0}^{n} \mathrm{P}(k)$. Hence, the values of $\mathrm{K}(n)$ can be calculated by the values of $\mathrm{P}(n)$. Particularly, $\mathrm{P}(0) = 1$, $\mathrm{P}(1) = 1$, $\mathrm{K}(0) = 1$, $\mathrm{K}(1) = 2$.

In the following, we give our main result of this paper.

Theorem 2.3. *The number of non-isomorphic zero-divisor semigroups corresponding to the graph $K_n - K_2$ (the complete graph K_n deleted an edge) is*

$$\mathrm{H}(n) = 4\sum_{i=0}^{n-2}\mathrm{K}(i) + 3\sum_{i=0}^{n-4}(n-3-i)\mathrm{K}(i) + \frac{1}{2}\sum_{i=0}^{n-5}(n-4-i)(n-3-i)\mathrm{K}(i).$$

Proof. Recall that we always suppose that $M_n = \{0, a_1, \ldots, a_n\}$ is a zero-divisor semigroup of the graph $K_n - K_2$, and $e = a_1 - a_2$ is the deleted edge. We can partition the set $\{a_i \mid 1 \le i \le n\}$ into the following three parts:

(1) $A = \{a_i \mid a_i^2 = 0\}$;

(2) $B = \{a_i \mid a_i^2 = a_i\}$;

(3) $C = \{a_i \mid a_i^2 = a_j,\ j \ne i\}$.

We have our discussions according to the possible values of a_1^2 and a_2^2. By symmetry, we only need to consider the following six cases: (1) $a_1 \in A$ and $a_2 \in A$; (2) $a_1 \in A$ and $a_2 \in B$; (3) $a_1 \in A$ and $a_2 \in C$; (4) $a_1 \in B$ and $a_2 \in B$; (5) $a_1 \in B$ and $a_2 \in C$; (6) $a_1 \in C$ and $a_2 \in C$. We note that for distinct cases, the corresponding semigroups are not isomorphic.

Case 1. Assume $a_1 \in A$ and $a_2 \in A$, so $a_1^2 = a_2^2 = 0$. In this case, $a_1a_2 \neq a_1, a_2$. Without loss of generality, we can assume $a_1a_2 = a_3$. So $a_3^2 = 0$, and $a_i^2 \in \{0, a_3, a_4, \ldots, a_n\}$, for all $i \geq 4$. We let $A_1 = \{a_i \mid a_i^2 = a_3, 4 \leq i \leq n\}$, and suppose that $|A_1| = k$. Without loss of generality, we can assume $a_4^2 = \cdots = a_{k+3}^2 = a_3$, for all $k + 4 \leq j \leq n$, if $a_j^2 = a_i$ for some $4 \leq i \leq k + 3$, then $a_i^2 = a_ia_j^2 = 0$, a contradiction. So, for all $k + 4 \leq j \leq n$, $a_j^2 \in \{0, a_{k+4}, \ldots, a_n\}$. In this case, $\{a_{k+4}, \ldots, a_n\}$ is a complete subgraph with $n - 3 - k$ vertices, and so we have $\mathrm{K}(n - 3 - k)$ non-isomorphic corresponding semigroups on $K_n - K_2$ by Lemma 2.2. Hence, in this case, we have

$$\sum_{k=0}^{n-3} \mathrm{K}(n - 3 - k)$$

non-isomorphic corresponding semigroups on $K_n - K_2$.

Case 2. Assume $a_1 \in A$ and $a_2 \in B$, so $a_1^2 = 0, a_2^2 = a_2$. In this case, if $a_1a_2 = a_2$, then $a_1a_2 = a_1^2a_2 = 0$, a contradiction. If $a_1a_2 = a_i$, for some $i \geq 3$, then $a_1a_2 = a_1a_2^2 = a_2a_i = 0$, a contradiction too. So, $a_1a_2 = a_1$ and for all $3 \leq i \leq n$, $a_i^2 \in \{0, a_3, \ldots, a_n\}$. We know that $\{a_3, \ldots, a_n\}$ is a complete subgraph with $n - 2$ vertices. By Lemma 2.2, in this case, we have $\mathrm{K}(n - 2)$ non-isomorphic corresponding semigroups on $K_n - K_2$.

Case 3. Assume $a_1 \in A$ and $a_2 \in C$, so $a_1^2 = 0, a_2^2 \neq 0, a_2$. In this case, without loss of generality, we can assume $a_2^2 = a_1$ or a_3.

Subcase 3.1. Assume $a_2^2 = a_1$, then $a_1a_2 \neq a_1, a_2$. We suppose that $a_1a_2 = a_3$, so $a_3^2 = 0$ and $a_i^2 \in \{0, a_3, a_4, \ldots, a_n\}$, for all $i \geq 4$. Similar to Case 1, we have

$$\sum_{k=0}^{n-3} \mathrm{K}(n-3-k)$$

non-isomorphic corresponding semigroups on $K_n - K_2$ in this subcase.

Subcase 3.2. Assume $a_2^2 = a_3$. Then $a_3^2 = a_2^2 a_3 = 0$, and $a_1 a_2 \neq a_1$, a_2. If $a_1 a_2 = a_3$, then $a_i^2 \in \{0, a_3, a_4, \ldots, a_n\}$, for all $i \geq 4$. Similar to Case 1, we have

$$\sum_{k=0}^{n-3} \mathrm{K}(n-3-k)$$

non-isomorphic corresponding semigroups on $K_n - K_2$. If $a_1 a_2 = a_4$, and so $a_4^2 = 0$. $a_i^2 \in \{0, a_3, a_4, \ldots, a_n\}$, for all $i \geq 5$. We let $A_1 = \{a_i \mid a_i^2 = a_3, 5 \leq i \leq n\}$, $A_2 = \{a_i \mid a_i^2 = a_4, 5 \leq i \leq n\}$, and suppose that $|A_1| = k$ and $|A_2| = l$. Without loss of generality, we can assume $a_5^2 = \cdots = a_{k+4}^2 = a_3$ and $a_{k+5}^2 = \cdots = a_{k+l+4}^2 = a_4$, then for all $k + l + 5 \leq j \leq n$, $a_j^2 \in \{0, a_{k+l+5}, \ldots, a_n\}$. Therefore, we have

$$\sum_{k+l=0}^{n-4} \mathrm{K}(n-4-(k+l))$$

non-isomorphic corresponding semigroups on $K_n - K_2$ by Lemma 2.2.

Case 4. Assume $a_1 \in B$ and $a_2 \in B$, so $a_1^2 = a_1$ and $a_2^2 = a_2$. In this case, $a_1 a_2 = a_1$ or a_2. By symmetry, we let $a_1 a_2 = a_1$. Then $\forall i$, $3 \leq i \leq n$, $a_i^2 \in \{0, a_3, \ldots, a_n\}$. By Lemma 2.2, we have $\mathrm{K}(n-2)$ non-isomorphic corresponding semigroups on $K_n - K_2$.

Case 5. Assume $a_1 \in B$ and $a_2 \in C$, so $a_1^2 = a_1$ and $a_2^2 \neq 0$, a_2. In this case, we claim that $a_1 a_2 = a_1$ or a_2 and $a_2^2 \neq a_i$, for all $i \geq 3$. So $a_2^2 = a_1$. In fact, if $a_1 a_2 = a_i$, for some $i \geq 3$, then $a_1 a_2 = a_1^2 a_2 = a_1 a_3$

$= 0$, a contradiction. If $a_2^2 = a_i$, for some $i \geq 3$, then $(a_1a_2)^2 = a_1^2a_2^2 = a_1a_i = 0$, so $a_1^2 = 0$ or $a_2^2 = 0$, a contradiction.

Subcase 5.1. Assume $a_1a_2 = a_1$. Then, for all $3 \leq i \leq n$, $a_i^2 \in \{0, a_3, \ldots, a_n\}$. By Lemma 2.2, in this subcase, we have $\mathrm{K}(n-2)$ non-isomorphic corresponding semigroups on $K_n - K_2$.

Subcase 5.2. Assume $a_1a_2 = a_2$. Then, $\forall i$, $3 \leq i \leq n$, $a_i^2 \in \{0, a_3, \ldots, a_n\}$. By Lemma 2.2, in this subcase, we have $\mathrm{K}(n-2)$ non-isomorphic corresponding semigroups on $K_n - K_2$.

Case 6. Assume $a_1 \in C$ and $a_2 \in C$. Then, we can assert that $a_1a_2 \neq a_1, a_2$ and $a_1^2 \neq a_2$, $a_2^2 \neq a_1$. In fact, if $a_1a_2 = a_1$, then $a_1a_2 = a_1a_2^2 = 0$ when $a_2^2 = a_i$, for some $i \geq 3$; $a_1a_2 = a_1a_2^2 = a_1^2$ when $a_2^2 = a_1$, this implies $a_1^2 = a_1$, a contradiction. Therefore, $a_1a_2 \neq a_1$. Similarly, $a_1a_2 \neq a_2$. On the other hand, if $a_1^2 = a_2$, then $a_2^2 = a_1^2a_2 = a_1a_i = 0$, a contradiction. And so, we only need to consider the following two subcases.

Subcase 6.1. Assume $a_1^2 = a_2^2 = a_3$. Then $a_3^2 = a_1^2a_3 = 0$. If $a_1a_2 = a_3$, then for all $4 \leq i \leq n$, $a_i^2 \in \{0, a_3, \ldots, a_n\}$. By Lemma 2.2, we have

$$\sum_{k=0}^{n-3} \mathrm{K}(n-3-k)$$

non-isomorphic corresponding semigroups on $K_n - K_2$. If $a_1a_2 = a_4$, then $a_4^2 = 0$, and then for all $5 \leq i \leq n$, $a_i^2 \in \{0, a_3, a_4, \ldots, a_n\}$. By Lemma 2.2, we have

$$\sum_{k+l=0}^{n-4} \mathrm{K}(n-4-(k+l))$$

non-isomorphic corresponding semigroups on $K_n - K_2$.

Subcase 6.2. Assume $a_1^2 = a_3$, $a_2^2 = a_4$. Then $a_3^2 = a_1^2 a_3 = 0$, and $a_4^2 = a_2^2 a_4 = 0$. If $a_1 a_2 = a_3$, then for all $5 \le i \le n$, $a_i^2 \in \{0, a_3, a_4, \ldots, a_n\}$. By Lemma 2.2, we have

$$\sum_{k+l=0}^{n-4} \mathrm{K}(n-4-(k+l))$$

non-isomorphic corresponding semigroups on $K_n - K_2$. If $a_1 a_2 = a_5$, then $a_5^2 = 0$, and then $\forall i$, $6 \le i \le n$, $a_i^2 \in \{0, a_3, a_4, a_5, \ldots, a_n\}$. By Lemma 2.2, we have

$$\sum_{k+l+m=0}^{n-5} \mathrm{K}(n-5-(k+l+m))$$

non-isomorphic corresponding semigroups on $K_n - K_2$.

Therefore, we have

$$\mathrm{H}(n) = 4\mathrm{K}(n-2) + 4\sum_{k=0}^{n-3} \mathrm{K}(n-3-k) + 3\sum_{k+l=0}^{n-4} \mathrm{K}(n-4-(k+l))$$

$$+ \sum_{k+l+m=0}^{n-5} \mathrm{K}(n-5-(k+l+m))$$

$$= 4\mathrm{K}(n-2) + 4\sum_{i=0}^{n-3} \mathrm{K}(i) + 3\sum_{i=0}^{n-4} (n-3-i)\mathrm{K}(i)$$

$$+ \frac{1}{2}\sum_{i=0}^{n-5} (n-4-i)(n-3-i)\mathrm{K}(i)$$

$$= 4\sum_{i=0}^{n-2} \mathrm{K}(i) + 3\sum_{i=0}^{n-4} (n-3-i)\mathrm{K}(i) + \frac{1}{2}\sum_{i=0}^{n-5} (n-4-i)(n-3-i)\mathrm{K}(i).$$

This completes our proof. □

Remark 2.4. From [6, (4.1.21)], we know that

$$\mathrm{P}(n) = \sum_{k=1}^{\infty}(-1)^{k-1}[\mathrm{P}(n - \frac{3k^2 - k}{2}) + \mathrm{P}(n - \frac{3k^2 + k}{2})].$$

By writing a programme, we can calculate the values of $\mathrm{P}(n)$, $\mathrm{K}(n)$, and $\mathrm{H}(n)$. We list, part values of $\mathrm{P}(n)$, $\mathrm{K}(n)$, and $\mathrm{H}(n)$ in the following table for $0 \leq n \leq 100$.

n	P(n)	K(n)	H(n)	n	P(n)	K(n)	H(n)
0	1	1	0	24	1575	7338	306667
1	1	2	0	25	1958	9296	414736
2	2	4	0	26	2436	11732	557115
3	3	7	12	27	3010	14742	743609
4	5	12	31	28	3718	18460	986618
5	7	19	69	29	4565	23025	1301650
6	11	30	142	30	5604	28629	1708141
7	15	45	271	31	6842	35471	2230241
8	22	67	494	32	8349	43820	2898002
9	30	97	860	33	10143	53963	3748531
10	42	139	1449	34	12310	66273	4827693
11	56	195	2368	35	14883	81156	6191819
12	77	272	3776	36	17977	99133	7910095
13	101	373	5886	37	21637	120770	10067062
14	135	508	9005	38	26015	146785	12765989
15	176	684	13536	39	31185	177970	16132438
16	231	915	20041	40	37338	215308	20319008
17	297	1212	29259	50	204226	1295971	174392770
18	385	1597	42188	60	966467	6639349	1193016410
19	490	2087	60128	70	4087968	30053954	6892184511
20	627	2714	84808	80	15796476	123223639	34891347958
21	792	3506	118454	90	56634173	465672549	158751402622
22	1002	4508	163981	100	190569292	1642992568	661085915682
23	1255	5763	225116	...	...	...	...

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